Lecture II
Quantum Leray–Hirsch and Analytic Continuations

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Classical Leray–Hirsch Theorem in topology:

Let $V \to S$ be a rank $r+1$ complex vector bundle, $X = P_S(V) \xrightarrow{p} S$ be the induced P^r bundle. Then

$$H(X) \cong p^* H(S)[h]/f_V(h)$$

as a ring isomorphism. Here $h = c_1(\mathscr{O}_X(1))$ and

$$f_V(h) = c_{r+1}(\mathscr{O}_X(1)) \otimes p^* V) = h^{r+1} + c_1(V)h^r + \cdots + c_{r+1}(V).$$

The fact $f_V(h) = 0$ in $H(X)$ follows from

$$0 \to \mathscr{O}_X(-1) \to p^* V \to Q \to 0.$$

How about $QH(X)$? (For X smooth projective.)

Let $e_i : \overline{M}_{g,n}(X, \beta) \to X$ be the evaluation maps. For $t = \sum t^\mu T_\mu \in H \equiv H(X)$, the Gromov–Witten potential

$$F_g^X(t) = \sum_{n,\beta} \frac{q^\beta}{n!} \langle t^n \rangle_{g,n,\beta}^X = \sum_{n \geq 0,\, \beta \in NE(X)} \frac{q^\beta}{n!} \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n e_i^* t$$

is a formal function in t^μ and q^β, with $\beta \in NE(X)$, the Mori cone. Let $g_{\mu\nu} = \langle T_\mu, T_\nu \rangle$ and $T^\mu = \sum g^{\mu\nu} T_\nu$,

$$\begin{aligned} T_\mu *_t T_\nu &= \sum_\kappa \frac{\partial^3 F_0^X}{\partial t^\mu \partial t^\nu \partial t^\kappa}(t) T^\kappa \\ &= \sum_{\kappa,\, n \geq 0,\, \beta \in NE(X)} \frac{q^\beta}{n!} \langle T_\mu, T_\nu, T_\kappa, t^n \rangle_{0,n+3,\beta}^X T^\kappa. \end{aligned}$$

The WDVV equations imply that $*_t$ is a family of associative products on H parameterized by $t \in H$.

Equivalently, a family (in $z \in \mathbb{C}^\times$) of flat Dubrovin connections

$$\nabla^z = d - \frac{1}{z} \sum_\mu dt^\mu \otimes T_\mu *_t$$

on the trivial tangent bundle $TH = H \times H$.

Goal of Quantum L–R: Describe $QH(X)$ using $\mathscr{D}$ modules:

$$QH(X) \quad \text{"} \cong \text{"} \quad p^* QH(S)[\hat{h}]/\widehat{f_V(h)}$$

with p^* and quantization $\hat{\bullet}$ suitably defined:

$$\widehat{f_V(h)} \quad \text{"} = \text{"} \quad \hat{h}^{r+1} + \widehat{c_1(V)}\hat{h}^r + \cdots + \widehat{c_{r+1}(V)}?$$

A baby example: $X = P^r$ ($S = \{\text{pt.}\}$)

Let $\ell = [P^1]$ be the line class. Then the small quantum cohomology ring, i.e. $t = 0$, is

$$QH_{small}(P^r) \cong \mathbb{C}[\hat{h}]/(\hat{h}^{r+1} - q^{\ell}).$$

Here $D = t^1 h \in H^2(P^r)$, $\hat{h} = z\partial_h = z\partial_{t^1}$ and the $\mathscr{D}^z$ module has basis $1, \hat{h}, \cdots, \hat{h}^r$ with Picard-Fuchs equation

$$\Box_{\ell} = (z\partial_h)^{r+1} - q^{\ell} = 0.$$

Such type of formula holds for **(semi-Fano) toric manifolds** (Lian-Liu-Yau, Givental, and Guest, Iritani for $\mathscr{D}$ module formulation) with the PF equation being replaced by the GKZ (Gelfand-Kapranov-Zelevinsky) system, and with certain mirror transformations.

A general framework to determine $g = 0$ GW invariants:

$$I \quad \text{(localization data)} \quad \Longrightarrow \quad J.$$

Let $\tau = \sum_\mu \tau^\mu T_\mu \in H$. J is the H-valued function:

$$J^X(\tau, z^{-1}) = 1 + \frac{\tau}{z} + \sum_{\beta \in NE(X), n, \mu} \frac{q^\beta}{n!} T_\mu \left\langle \frac{T^\mu}{z(z-\psi)}, \tau, \cdots, \tau \right\rangle_{0,n+1,\beta}.$$

The variable z is responsible for the weight of $\mathbb{C}^\times$ action on he domain curve which keeps track on the degree of the $\psi = \psi_1$ class. (Here $\psi_i = c_1(L_i)$ with $L_i = \sigma_i^* \omega_{\mathscr{C}/\mathscr{M}}$.)

Witten's **DE + SE + TRR** (dilaton, string, topological recursion relation) in 2D gravity $\Leftrightarrow$ Givental's **symplectic reformulation**.

Let $\mathcal{H} := H[z, z^{-1}]]$, $\mathcal{H}_+ := H[z]$ and $\mathcal{H}_- := z^{-1}H[[z^{-1}]]$. Then $\mathcal{H} \cong T^*\mathcal{H}_+$ is naturally a symplectic space (formal loops).

Let $F_0(\mathbf{t})$ be the generating function of all descendent invariants, where

$$\mathbf{t} = \sum t_k^\mu z^k T_\mu \in \mathcal{H}_+.$$

The one form dF_0 gives a section of $\pi : \mathcal{H} \to \mathcal{H}_+$.

Givental's Lagrangian cone at $\mathbf{t} = 0$ is

$$\mathcal{L} = \text{graph of } dF_0 \qquad (\text{DE} \Longrightarrow \text{cone})$$

with $-zJ(\tau, -z^{-1})$ a partial section over $-z1 + H$.

Let $R = \mathbb{C}[\widehat{NE(X)}]$ and $a = \sum q^\beta a_\beta(z) \in R\{z\}$ if $a_\beta(z) \in \mathbb{C}[z]$.

Lemma (Givental)

$z\nabla J = (z\partial_\mu J^\nu)$ forms a matrix whose column vectors $z\partial_\mu J(\tau)$ generates the tangent space L_τ of $\mathcal{L}$ as an $R\{z\}$-module.

In fact, TRR $\Longrightarrow z\nabla J$ is the fundamental solution matrix of ∇^z, i.e. we have the quantum differential equation (QDE)

$$z\partial_\mu \, z\partial_\nu J = \sum \tilde{C}^\kappa_{\mu\nu}(\tau, q) \, z\partial_\kappa J,$$

where

$$T_\mu *_\tau T_\nu = \sum \tilde{C}^\kappa_{\mu\nu} \, T_\kappa.$$

That is,

> $QH(X)$ **is a cyclic holonomic $\mathscr{D}^z$ module generated by J.**

The rank is $N = \dim H$, with

$$z\partial_\mu J = T_\mu + O(z^{-1}).$$

Let $\bar{p}: X \to S$ be a smooth toric bundle with fiber divisor $D = \sum t^i D_i$. $H(X)$ is a free over $H(S)$ with finite generators $\{D^e := \prod_i D_i^{e_i}\}_{e \in \Lambda}$. Let $\bar{t} := \sum_s \bar{t}^s \bar{T}_s \in H(S)$. $H(X)$ has basis

$$\{T_{\mathbf{e}} = T_{(s,e)} = \bar{T}_s D^e\}_{\mathbf{e} \in \Lambda^+}.$$

Denote by $\partial_{\bar{T}_s} \equiv \partial_{\bar{t}^s}$ the $\bar{T}_s$ directional derivative on $H(S)$, $\partial^{\mathbf{e}} = \partial^{(s,e)} := \partial_{\bar{t}^s} \prod_i \partial_{t^i}^{e_i}$, and the **naive quantization**

$$\hat{T}_e \equiv \partial^{z\mathbf{e}} \equiv \partial^{z(s,e)} := z\partial_{\bar{t}^s} \prod_i z\partial_{t^i}^{e_i} = z^{|e|+1} \partial^{(s,e)}.$$

The $T_{\mathbf{e}}$ directional derivative on $H(X)$ is $\partial_{\mathbf{e}} = \partial_{T_{\mathbf{e}}}$.

$$\partial^{z\mathbf{e}} \quad \text{(higher order)} \quad \text{and} \quad z\partial_{\mathbf{e}} \quad \text{(first order)}$$

look different, but they are closely related.

Let $\bar{p} : X \to S$ be a **split** toric bundle quotient from

$$V = \bigoplus \mathscr{L}_\rho \to S.$$

The hypergeometric modification of J^S by the $\bar{p}$-fibration takes the form

$$I^X(\bar{t}, D, z, z^{-1}) := \sum_{\beta \in NE(X)} q^\beta e^{\frac{D}{z} + (D.\beta)} I_\beta^{X/S}(z, z^{-1}) J_{\beta_S}^S(\bar{t}, z^{-1}),$$

where

$$I_\beta^{X/S} = \prod_{\rho \in \triangle_1} \frac{1}{\prod_{m=1}^{(D_\rho + \mathscr{L}_\rho).\beta} (D_\rho + \mathscr{L}_\rho + mz)}$$

comes from **fiber localization**, and the product is *directed* in the sense that it appears in the numerator when $(D_\rho + \mathscr{L}_\rho).\beta \leq -1$.

In general, I^X is only an approximation of J^X since positive z powers may occur in I^X.

Also the approximation is valid only in some directions since I is defined only on **the thin subspace**

$$\hat{t} := \bar{t} + D \in H(S) \oplus \bigoplus_i \mathbb{C}D_i \subset H(X).$$

Theorem (J. Brown 2009)

$(-z)I^X(\hat{t}, -z)$ *lies in the Lagrangian cone* $\mathcal{L}$.

Definition (GMT)

For each $\hat{t}$, say $zI(\hat{t})$ lies in L_τ of $\mathcal{L}$. The correspondence

$$\hat{t} \mapsto \tau(\hat{t}) \in H(X) \otimes R$$

is called the *generalized mirror transformation*.

So $I = QJ$ for a first order operator Q. However, we need the reverse direction to determine J from I, at least in some range:

Theorem (BF/GMT, LLW)

(1) *The GMT:* $\tau = \tau(\hat{t})$ *satisfies* $\tau(\hat{t}, q = 0) = \hat{t}$.

(2) *Under the basis* $\{T_{\mathbf{e}}\}_{\mathbf{e}\in\Lambda^+}$, *there exists an invertible* $N \times N$ *matrix-valued formal series* $B(\tau, z)$, *the Birkhoff factorization, such that*

$$\left(\partial^{z\mathbf{e}} I(\hat{t}, z, z^{-1})\right) = \left(z\nabla J(\tau, z^{-1})\right) B(\tau, z),$$

where $\partial^{z\mathbf{e}} I$ *and* $z\partial_{\mathbf{e}} J$ *are column vectors. The first column vectors are* I *and* J *respectively (string equation).*

(3) *Both* τ, B, *and hence* J *are effectively computable from* I. *In fact*

$$J(\tau) = P(z) I(\hat{t})$$

for an inductively determined (in $NE(S)$*) higher order operator* P *which eliminates all positive* z *powers of the RHS. Finally,*

$$\tau(\hat{t}) = z^{-1}\text{-coefficient of } PI.$$

Unfortunately, this does not provide any conceptual behavior or **analytic properties** of τ, B or J in $\hat{t} = \bar{t} + D$.

In particular, this does not provide a solution to the Quantum Leray–Hirsch problem as proposed.

Target problem: Consider an ordinary P^r flop

$$\begin{array}{ccc} Z \subset X & \overset{f}{\dashrightarrow} & Z' \subset X' \\ & {\scriptstyle\psi}\searrow \qquad \swarrow{\scriptstyle\psi'} & \\ & S \subset \bar{X} & \end{array}$$

where we know

$$\mathscr{F} = [\bar{\Gamma}_f] \in A(X \times X')$$

gives $H(X) \cong H(X')$ as groups and Hodge structures, but not the cup product.

Consider the local model with split bundle data (S, F, F'):

$$F = \bigoplus_{i=0}^{r} L_i \quad \text{and} \quad F' = \bigoplus_{i=0}^{r} L_i'.$$

Then $\bar{\psi} : Z = P_S(F) \to S$ with

$$N = N_{Z/X} = \bar{\psi}^* F' \otimes \mathscr{O}_Z(-1),$$

and

$$\bar{p} : X = P_Z(N \oplus \mathscr{O}) \xrightarrow{p} Z \xrightarrow{\bar{\psi}} S$$

is a **double projective bundle**. Similarly for Z', N' and X''.
The flop is then

$$X = P_Z(N \oplus \mathscr{O}) \dashrightarrow X' = P_{Z'}(N' \oplus \mathscr{O}).$$

Leray–Hirsch $\Longrightarrow$
for h, ξ being the *relative hyperplane classes*,

$$H(X) = H(S)[h,\xi]/(f_F, f_{N\oplus\mathscr{O}}),$$

where (**notice the sign of** h)

$$f_F = \prod_{i=0}^{r} a_i := \prod (h + L_i),$$

$$f_{N\oplus\mathscr{O}} = b_{r+1}\prod_{i=0}^{r} b_i := \xi \prod (\xi - h + L_i').$$

$\mathscr{F} : H(X) \cong H(X')$ as groups is easy: for $\bar{t} \in H(S)$,

$$\mathscr{F}\bar{t}h^i\xi^j = \bar{t}(\mathscr{F}h)^i(\mathscr{F}\xi)^j = \bar{t}(\xi' - h')^i\xi'^j, \qquad i \le r.$$

(For $j > 0$ this trivially holds for all i.)

Theorem (LLW-II 2010, split flop case, AG 2016)

$\mathscr{F}$ induces an isomorphism of quantum rings $QH(X) \cong QH(X')$ under analytic continuations in the Kähler moduli.

Let γ, ℓ be the fiber line classes in $X \to Z \to S$. Then

$$\mathscr{F}\gamma = \gamma' + \ell',$$

but for the **extremal rays**:

$$\mathscr{F}\ell = -\ell' \notin NE(X'),$$

so analytic continuations are necessary.

[Li–Ruan 2000] ($r = 1$, $\dim X = 3$),
[LLW 2006] (simple P^r flop in any dimension, $S = \text{pt}$),
[LLW 2008] (simple flop, any $g \geq 0$).

Any $\beta \in A_1(X)$ is of the form

$$\beta = \beta_S + d\ell + d_2\gamma$$

where $\beta_S \in A_1(S)$ is identified with its **canonical lift** in $A_1(X)$ with $(\beta_S.h) = 0 = (\beta_S.\xi)$. h, ξ are dual to ℓ, γ hence

$$\beta.h = d, \qquad \beta.\xi = d_2.$$

To construct the $\mathscr{D}^z$ module $QH(X)$:

Proposition (Picard–Fuchs system on X/S)

The **fiber directions** *are handled by* $\Box_\ell I^X = 0$ *and* $\Box_\gamma I^X = 0$*, where*

$$\Box_\ell = \prod_{j=0}^r z\partial_{a_j} - q^\ell e^{t^1} \prod_{j=0}^r z\partial_{b_j}, \quad \Box_\gamma = z\partial_\xi \prod_{j=0}^r z\partial_{b_j} - q^\gamma e^{t^2}.$$

Moreover, we have $\mathscr{F}$*-invariance*

$$\mathscr{F}\langle \Box_\ell^X, \Box_\gamma^X \rangle \cong \langle \Box_{\ell'}^{X'}, \Box_{\gamma'}^{X'} \rangle.$$

I^X satisfies the two equations by a direct computation from

$$I_\beta^{X/S} = \prod_{i=0}^{r} \frac{1}{\prod\limits_{m=1}^{\beta.a_i} (a_i + mz)} \prod_{i=0}^{r} \frac{1}{\prod\limits_{m=1}^{\beta.b_i} (b_i + mz)} \frac{1}{\prod\limits_{m=1}^{\beta.\xi} (\xi + mz)}.$$

The $\mathscr{F}$-invariance of Picard-Fuchs ideals: Since

$$\mathscr{F} a_j = \mathscr{F}(h + L_i) = \xi' - h' + L_i = b'_j$$

and $\mathscr{F} b_j = a'_j$ for $0 \le j \le r$. It is clear that

$$\mathscr{F} \Box_\ell = -q^{-\ell'} e^{t^1} \Box_{\ell'},$$

and

$$\mathscr{F} \Box_\gamma = z\partial_{\xi'} \prod_{j=0}^{r} z\partial_{a'_j} - q^{\gamma' + \ell'} e^{t^2} = z\partial_{\xi'} \Box_{\ell'} + q^{\ell'} e^{-t^1} \Box_{\gamma'}.$$

To handle the **base directions**, we need to lift the QDE on S to X. This requires to **lift the Mori cone** first.

Lemma (I-minimal lift = generic lift)

(i) Given a primitive $\beta_S \in NE(S)$, $\beta \in NE(X)$ if and only if

$$d \geq -\mu \quad \textit{and} \quad d_2 \geq -\nu,$$

where

$$\mu = \max_i \{\beta_S . L_i\}, \quad \mu' = \max_i \{\beta_S . L_i'\}, \quad \nu = \max\{\mu + \mu', 0\}.$$

(ii) For general β_S, these numerical condition defines

$$NE^I(X) \subset NE(X).$$

The minimal one β_S^I is called the I-minimal lift.

Observation: $I_\beta^{X/S}$ is non-trivial only if $\beta \in NE^I(X)$.

Definition (Operator $D_\beta(z)$)

For any one cycle $\beta \in A_1(X)$, we define for $0 \le i \le r$,

$$n_i(\beta) := -\beta.(h + L_i),$$
$$n'_i(\beta) := -\beta.(\xi - h + L'_i), \quad n'_{r+1}(\beta) := -\beta.\xi,$$

It is **admissible** if these are all ≥ 0. Then $D_\beta(z) := D^A_\beta D^B_\beta D^C_\beta$:

$$D^A_\beta = \prod_{i=0}^{r} \prod_{m=0}^{n_i(\beta)-1} (z\partial_{h+L_i} - mz),$$
$$D^B_\beta = \prod_{i=0}^{r} \prod_{m=0}^{n'_i(\beta)-1} (z\partial_{\xi-h+L'_i} - mz), \quad D^C_\beta = \prod_{m=0}^{n'_{r+1}(\beta)-1} (z\partial_\xi - mz).$$

Key point: For $\bar{\beta} \in NE(S)$, the I-minimal lift $\bar{\beta}^I$ is admissible.

Theorem (Quantum Leray–Hirsch)

(1) *(I-Lifting) The QDE on $QH(S)$ can be lifted to $H(X)$ as*

$$z\partial_i\, z\partial_j I = \sum_{k,\bar\beta\in NE(S)} q^{\bar\beta^I} e^{(D.\bar\beta^I)} \bar C^k_{ij,\bar\beta}(\bar t)\, z\partial_k D_{\bar\beta^I}(z) I,$$

with $D_{\bar\beta^I}(z)$ unique modulo the Picard–Fuchs system.

(2) *Together with the Picard–Fuchs $\Box_\ell$ and $\Box_\gamma$, they determine a first order matrix system under the naive quantization basis:*

$$z\partial_a(\partial^{z\mathbf{e}} I) = (\partial^{z\mathbf{e}} I) C_a(z,q), \qquad \text{where } t^a = t^1, t^2 \text{ or } \bar t^i.$$

(3) *For $\bar\beta \in NE(S)$, its coefficients in C_a are polynomial in $q^\gamma e^{t^2}$, $q^\ell e^{t^1}$ and $\mathbf{f}(q^\ell e^{t^1})$, and formal in $\bar t$. Here*

$$\mathbf{f}(q) := q/(1-(-1)^{r+1}q)$$

satisfies $\mathbf{f}(q) + \mathbf{f}(q^{-1}) = (-1)^r$. (Origin of functional eqn.)

(4) *The system is $\mathscr{F}$-invariant, though in general $\mathscr{F}\bar\beta^I \neq \bar\beta^{I'}$.*

Idea: (1) Write $\bar{t} = \sum \bar{t}^i \bar{T}_i$ and $\bar{C}_{ij}^k(\bar{t}, \bar{q}) = \sum_{\bar{\beta} \in NE(S)} \bar{C}_{ij,\bar{\beta}}^k(\bar{t})\, q^{\bar{\beta}}$, then

$$z\partial_i z\partial_j J_{\bar{\beta}}^S = \sum_{k, \bar{\beta}_1 \in NE(S)} \bar{C}_{ij,\bar{\beta}_1}^k \, z\partial_k J_{\bar{\beta}-\bar{\beta}_1}^S.$$

Since i, j are in the base $H(S)$ directions,

$$\begin{aligned} z\partial_i z\partial_j I &= \sum_{\beta} q^{\beta} e^{\frac{D}{z} + (D.\beta)} I_{\beta}^{X/S} z\partial_i z\partial_j J_{\bar{\beta}}^S \\ &= \sum_{k, \bar{\beta}_1} q^{\bar{\beta}_1^I} e^{D.\bar{\beta}_1^I} \bar{C}_{ij,\bar{\beta}_1}^k z\partial_k \sum_{\beta} q^{\beta - \bar{\beta}_1^I} e^{\frac{D}{z} + D.(\beta - \bar{\beta}_1^I)} I_{\beta}^{X/S} J_{\bar{\beta}-\bar{\beta}_1}^S. \end{aligned}$$

The key is to rewrite $I_{\beta}^{X/S} \cdots$ as

$$D_{\bar{\beta}^I}(z)(I_{\beta - \bar{\beta}_1^I}^{X/S} \cdots).$$

(2), (3) are by induction on β_S. (4) follows from the equivalence of the Picard–Fuchs ideals.

Ideas involved in the proof of (2) and (3).

The Picard–Fuchs system generated by $\Box_\ell$ and $\Box_\gamma$ is a **perturbation** of the Picard–Fuchs (hypergeometric) system associated to the (toric) fiber **by operators in base divisors**.

The fiberwise toric case is a GKZ system, which by the theorem of Gelfand–Kapranov–Zelevinsky is a holonomic system of rank $(r+1)(r+2)$, the dimension of cohomology space of a fiber. It is also known that the GKZ system admits a Gröbner basis reduction to the holonomic system.

We apply this result in the following manner: we would like to construct a $\mathscr{D}$ module with basis

$$\partial^{z\mathbf{e}} I, \qquad \mathbf{e} \in \Lambda^+.$$

We apply $z\partial_{t^1}, z\partial_{t^2}$ and first order operators $z\partial_i$'s to this selected basis.

Notice that

$$\Box_\ell = (1 - (-1)^{r+1} q^\ell e^{t^1})(z\partial_{t^1})^{r+1} + \cdots,$$
$$\Box_\gamma = (z\partial_{t^2})^{r+2} + \cdots.$$

This is where **f** appears. The Gröbner basis reduction allows one to reduce the differentiation order in $z\partial_{t^1}$ and $z\partial_{t^2}$ to smaller one $\partial^{z\mathbf{e}} I$'s with $\mathbf{e} \in \Lambda^+$. In the process higher order differentiation in $z\partial_i$'s will be introduced.

Using (1), the I-lifting, the differentiation in the base direction with order > 1 can be reduced to one by introducing more terms with strictly larger effective classes in $NE(S)$.

A careful induction will conclude the proof. In fact in the current special case coming from ordinary flops, neither the GKZ theorem nor the Gröbner basis were needed.

All the analytic continuations are reduced to the one on **f**. $\Box$

Finally we will construct a gauge transformation B to eliminate all the z dependence of C_a in the $\mathscr{F}$-invariant system

$$z\partial_a(\partial^{z\mathbf{e}}I) = (\partial^{z\mathbf{e}}I)C_a,, \qquad t^a = t^1, t^2 \text{ or } \bar{t}^i. \tag{1}$$

B is nothing more than the **Birkhoff factorization matrix**

$$\partial^{z\mathbf{e}}I(\hat{t}; z, z^{-1}) = (z\nabla J)(\tau; z^{-1})B(\tau; z) \tag{2}$$

valid at the **generalized mirror point** $\tau = \tau(\hat{t})$.
Substituting (2) into (1), we get

$$z\partial_a(\nabla J)B + z(\nabla J)\partial_a B = (\nabla J)BC_a,$$

hence

$$z\partial_a(\nabla J) = (\nabla J)(-z\partial_a B + BC_a)B^{-1} =: (\nabla J)\tilde{C}_a. \tag{3}$$

We must notice the subtlety in the meaning of $\tilde{C}_a(\hat{t})$.

Let $\tau = \sum \tau^\mu T_\mu$. Write the QDE as

$$z\partial_\mu(\nabla J)(\tau) = (\nabla J)(\tau)\tilde{C}_\mu(\tau),$$

then

$$z\partial_a(\nabla J) = \sum_\mu \frac{\partial \tau^\mu}{\partial t^a} z\partial_\mu(\nabla J) = (\nabla J)\sum_\mu \tilde{C}_\mu \frac{\partial \tau^\mu}{\partial t^a},$$

hence

$$\tilde{C}_a(\hat{t}) \equiv \sum_\mu \tilde{C}_\mu(\tau(\hat{t}))\frac{\partial \tau^\mu}{\partial t^a}(\hat{t}). \tag{4}$$

In particular, $\tilde{C}_a$ is independent of z. And (3) is equivalent to

$$\tilde{C}_a = B_0 C_{a;0} B_0^{-1} \tag{5}$$

($B_0^{-1} := (B^{-1})_0$, coefficient matrix of z^0) and the **cancellation equation**

$$z\partial_a B = BC_a - B_0 C_{a;0} B_0^{-1} B. \tag{6}$$

Analyze $B = B(z)$ by induction on $w := (\bar{\beta}, d_2) \in W$. The initial condition is the extremal ray case $B_{w=(0,0)} = \mathrm{Id}$.
Suppose that $B_{w'}$ satisfies $\mathscr{F} B_{w'} = B'_{w'}$ for all $w' < w$. Then

$$z\partial_a B_w = \sum_{w_1+w_2=w} B_{w_1} C_{a;w_2} - \sum_{w_1+w_2+w_3+w_4=w} B_{w_1,0} C_{a;w_2,0} B^{-1}_{w_3,0} B_{w_4}.$$

Write $B_w = \sum_{j=0}^{n(w)} B_{w,j}\, z^j$. Then in the RHS all the B terms have strictly smaller degree than w except

$$B_w C_{a;(0,0)} - C_{a;(0,0)} B_w + B_{w,0} C_{a;(0,0)} - C_{a;(0,0)} B^{-1}_{w,0}$$

which has maximal z degree $\leq n(w)$. By descending induction on j, the z degree, we get

$$\partial_a (\mathscr{F} B_{w,j} - B'_{w,j}) = 0.$$

The functions involved are all formal in $\bar{t}$ and analytic in t^1, t^2, and without constant term ($B_{w=(0,0)} = \mathrm{Id}$). Hence $\mathscr{F} B_{w,j} = B'_{w,j}$.

We have proved that for any $\hat{t} = \bar{t} + D \in H(S) \oplus \mathbb{C}h \oplus \mathbb{C}\xi$,

$$\mathscr{F} B(\tau(\hat{t})) \cong B'(\tau'(\hat{t})),$$

hence the **$\mathscr{F}$-invariance of** $\tilde{C}_a(\hat{t}) = B_0 C_{a;0} B_0^{-1}$: Explicitly

$$\tilde{C}_{a\nu}^{\kappa} = \sum_{n \geq 0, \mu} \frac{q^{\beta}}{n!} \frac{\partial \tau^{\mu}(\hat{t})}{\partial t^a} \langle T_{\mu}, T_{\nu}, T^{\kappa}, \tau(\hat{t})^n \rangle_{\beta}.$$

The case $T_\nu = 1$ leads to non-trivial invariants only for 3-point classical invariant ($n = 0$) and $\beta = 0$, and also $\mu = \kappa$. Since κ is arbitrary, we have thus proved the $\mathscr{F}$-invariance of $\partial_a \tau$. Then

$$\partial_a(\mathscr{F}\tau - \tau') = \mathscr{F}\partial_a \tau - \partial_a \tau' = 0.$$

Again since $\tau(\hat{t}) = \hat{t}$ for $(\bar{\beta}, d_2) = (0, 0)$, this proves

$$\mathscr{F}\tau = \tau',$$

which is an **analytic continuation in the q^ℓ variable**. □

Conclusion of proof of Q-inv

Since

$$\mathscr{F}\tau \cong \tau', \qquad \mathscr{F}B(\tau) \cong B'(\tau'),$$

we get

$$\mathscr{F}J(\tau(\hat{t})).\xi \cong J'(\tau'(\mathscr{F}\hat{t})).\xi'$$

and then by Mori cone induction it is not hard to get

$$\mathscr{F}J(\hat{t}).\xi \cong J'(\hat{t}).\xi'.$$

This simply means

$$\mathscr{F}\langle \hat{t}^n, \psi^k 1a\rangle^X \cong \langle \mathscr{F}\hat{t}^n, \psi^k \xi \mathscr{F}a\rangle^{X'}.$$

The general case of f-special type invariants with more descendents and with general insertions than $\hat{t}$ then follows by the **divisorial reconstruction**. **QED**

Example I:

Hirzebruch surface $X = \Sigma_{-1}$. This is the P^1 bundle over $S = P^1$ associated to $V = \mathscr{O} \oplus \mathscr{O}(1)$.

Write $H(S) = H(P^1) = \mathbb{C}[p]/(p^2)$.

$$H = H(X) = H(S)[h]/\langle h(h+p)\rangle$$

has dimension $N = 4$. Consider the basis $\{T_i \mid 1 \leq i \leq 4\}$ by

$$1,\ h,\ p,\ hp.$$

Denote $q = q^{\ell} e^{t}, \bar{q} = q^{b} e^{\bar{t}}$, where $b = [S] \cong [P^1]$. Then

$$\Box_{\ell} = (z\partial_h)(z\partial_{h+p}) - q.$$

It leads to the reduction procedure in $z\partial_h$:

$$(z\partial_h)^2 = q - (z\partial_h)(z\partial_p). \tag{7}$$

Since $H(S) = \mathbb{C}1 \oplus \mathbb{C}p$, the small and big quantum rings coincide. It is easy to compute its QDE:

$$z\partial_p(z\partial_1, z\partial_p) = (z\partial_1, z\partial_p)\begin{pmatrix} 0 & \bar{q} \\ 1 & 0 \end{pmatrix}.$$

Since $b^I = b - \ell$, we get $D_{b^I}(z) = z\partial_h$. We get the lifting of the QDE to be

$$(z\partial_p)^2 = \bar{q}q^{-1} z\partial_h. \tag{8}$$

By (7) and (8), we calculate the matrix C_{t^a} of the action of $z\partial_{t^a} = z\partial_h$ or $z\partial_p$ on $\hat{T}_i$ as $z\partial_{t^a}\hat{T}_j = \sum_k C^k_{t^a j}(z)\hat{T}_k$ modulo I^X. Then

$$C_h = \begin{bmatrix} 0 & q & 0 & -\bar{q} \\ 1 & 0 & 0 & z\bar{q}q^{-1} \\ 0 & 0 & 0 & q \\ 0 & -1 & 1 & \bar{q}q^{-1} \end{bmatrix},$$

$$C_p = \begin{bmatrix} 0 & 0 & 0 & \bar{q} \\ 0 & 0 & \bar{q}q^{-1} & -z\bar{q}q^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\bar{q}q^{-1} \end{bmatrix}.$$

We solve B from C_h and C_p by the recursive equation (27):
$B_{2,4} = -\bar{q}q^{-1}$,

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\bar{q}q^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first column implies that in $J = PI$, $P(z) = 1$ and $\tau(\hat{t}) = \hat{t}$. The full matrix system requires basis in all directions which uses the full matrix B and non-trivial BF is required.

$$B = I_4 - \bar{q}q^{-1}e_{2,4}, \qquad B^{-1} = I_4 + \bar{q}q^{-1}e_{2,4}.$$

From this we get $\tilde{C}_{t^a} = B_0 C_{t^a;0} B_0^{-1}$:

$$\tilde{C}_h = \begin{bmatrix} 0 & q & 0 & 0 \\ 1 & \bar{q}q^{-1} & -\bar{q}q^{-1} & 0 \\ 0 & 0 & 0 & q \\ 0 & -1 & 1 & 0 \end{bmatrix}, \quad \tilde{C}_p = \begin{bmatrix} 0 & 0 & 0 & \bar{q} \\ 0 & -\bar{q}q^{-1} & \bar{q}q^{-1} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

By setting $\hat{t} = 0$, we get $q = q^\ell$ and $\bar{q} = q^b$, and we can read out the 3-point invariants. E.g. for the entries at $(2, 3)$:

$$\tilde{C}^3_{h2} = \langle T_2, T_2, T^3 \rangle = \langle h, h, p^* \rangle = \delta_h \langle h, h \rangle = -q^{-\ell} q^b,$$
$$\tilde{C}^3_{p2} = \langle T_3, T_2, T^3 \rangle = \langle p, h, p^* \rangle = \delta_p \langle h, h \rangle = q^{-\ell} q^b.$$

This coincides with more classical methods.

Example II:

(Non-trivial BF/GMT.) A P^1 flop $f : X \dashrightarrow X'$ with local data $(S, F, F') = (P^1, \mathscr{O} \oplus \mathscr{O}, \mathscr{O} \oplus \mathscr{O}(1))$.

Write $H(S) = \mathbb{C}[p]/(p^2)$ with Chern polynomials

$$f_F(h) = h^2, \qquad f_{N \oplus \mathscr{O}}(\xi) = \xi(\xi - h)(\xi - h + p).$$

Then $H = H(X) = H(S)[h, \xi]/(f_F, f_{N \oplus \mathscr{O}})$ has dimension $N = 12$ with basis $\{T_i \mid 0 \le i \le 11\}$ being

$$1,\ h,\ \xi,\ p,\ h\xi,\ hp,\ \xi^2,\ \xi p,\ h\xi^2,\ h\xi p,\ \xi^2 p,\ h\xi^2 p.$$

Denote by $q_1 = q^{\ell} e^{t^1}$, $q_2 = q^{\gamma} e^{t^2}$, $\bar{q} = q^b e^{t^3}$, where $b = [S] \cong [P^1]$, and $\mathbf{f} = \mathbf{f}(q_1)$. The Picard-Fuchs operators are

$$\Box_\ell = (z\partial_h)^2 - q_1 z\partial_{\xi - h}\, z\partial_{\xi - h + p},$$
$$\Box_\gamma = z\partial_\xi\, z\partial_{\xi - h}\, z\partial_{\xi - h + p} - q_2.$$

They lead to a Grobner basis:

$$(z\partial_h)^2 = \mathbf{f}(z\partial_\xi)^2 - \mathbf{f}\, z\partial_p\, z\partial_h + \mathbf{f}\, z\partial_p\, z\partial_\xi - 2\mathbf{f}\, z\partial_h\, z\partial_\xi \tag{9}$$

$$(z\partial_\xi)^3 = q_2(1-q_1) - z\partial_p(z\partial_\xi)^2 + 2z\partial_h(z\partial_\xi)^2 + z\partial_p\, z\partial_h\, z\partial_\xi. \tag{10}$$

$H(S) = \mathbb{C}1 \oplus \mathbb{C}p$ has only small parameters with QDE

$$z\partial_p(z\partial_1, z\partial_p) = (z\partial_1, z\partial_p)\begin{pmatrix} 0 & \bar{q} \\ 1 & 0 \end{pmatrix}.$$

The real difference from the previous $((0,0),(0,-1))$ case starts with the lifting of this QDE. Now $b^I = b - \gamma$, we get $D_b = z\partial_\xi\, z\partial_{\xi-h}$, and the lifting becomes

$$(z\partial_p)^2 = \bar{q}q_2^{-1}\, z\partial_\xi\, z\partial_{\xi-h}. \tag{11}$$

By (9), (10) and (11), we calculate C_a in $z\partial_a \hat{T}_j = \sum_k C_{aj}^k(z)\hat{T}_k$ modulo I^X. Let $q^* = \bar{q}q_2^{-1}$ be the chosen admissible lift and set $\mathbf{g} = \mathbf{f}(q^*)$, $A = q_2 - q_1q_2$, $S = q_2 + q_1q_2$. We get

$$
C_h = \begin{bmatrix}
 & & & & q_1q_2 & \mathbf{f}\, q_2q^* & & & zq_1q_2 & & & \\
1 & & & & & & & & & & & \\
 & & & & & & & & q_1q_2 & & & \\
 & & & & & & & & & q_1q_2 & & zq_1q_2 \\
 & -2\mathbf{f} & 1 & & & z\mathbf{f}\, q^* & & & & & & \\
 & -\mathbf{f} & & 1 & & & & & & & & \\
 & \mathbf{f} & & & & -z\mathbf{f}\, q^* & & & & & & \\
 & \mathbf{f} & & & & & & & & & & q_1q_2 \\
 & & & & & & 1 & & & & & \\
 & & & & & \mathbf{f}(q^*-2) & & 1 & & & & \\
 & & & & & \mathbf{f}(1-q^*) & & & & & & \\
 & & & & & & & & & & 1 &
\end{bmatrix},
$$

$$
C_\xi = \begin{bmatrix}
 & & & & & & A & & zq_1q_2 & & zA\mathbf{g} & z^2q_1q_2\mathbf{g} \\
 & & & & & & & & A & & & zA\mathbf{g} \\
1 & & & & & & & & 2q_1q_2 & & -q_2\mathbf{g} & zq_1q_2\mathbf{g} \\
 & & & & & & & & q_1q_2 & & A(1+\mathbf{g}) & zq_1q_2(1+2\mathbf{g}) \\
 & 1 & & & & & & & & & z^2\mathbf{g} & -q_2q^*(1+\mathbf{g}) \\
 & & & & & & & & & & & A(1+\mathbf{g}) \\
 & & 1 & & & & & & & & -z^2\mathbf{g} & \\
 & & & 1 & & & & & & & & q_1q_2(2+\mathbf{g}) \\
 & & & & 1 & & 2 & & & & z\mathbf{g} & -z^2\mathbf{g} \\
 & & & & & 1 & 1 & & & & 2z\mathbf{g} & \\
 & & & & & & -1 & 1 & & & -2z\mathbf{g} & \\
 & & & & & & & & -1 & 1 & 2+\mathbf{g} & -2z\mathbf{g}
\end{bmatrix},
$$

and $C_p =$

$$
\begin{bmatrix}
 & & & & & -q_1q_2q^* & & Aq^* & & zq_1q_2q^* & z(q_1q_2q^* - A\mathbf{g}) & -z^2q_1q_2\mathbf{g} \\
 & & & & & & & & & Aq^* & Aq^* & -zA\mathbf{g} \\
 & & & & & & & & & q_1q_2q^* & (S - q_1q_2q^*)\mathbf{g} & -zq_1q_2\mathbf{g} \\
1 & & & & & & & & & q_1q_2q^* & q_1q_2q^* - A\mathbf{g} & -2zq_1q_2\mathbf{g} \\
 & & & -q^* & & & & zq^* & & & -z^2\mathbf{g} & (A + q_1q_2q^*)\mathbf{g} \\
 & 1 & & & & & & & & & & -A\mathbf{g} \\
 & & & q^* & & & & -zq^* & & & z^2\mathbf{g} & q_1q_2q^* \\
 & & 1 & & & & & & & & & -q_1q_2\mathbf{g} \\
 & & & & & q^* & & q^* & & -zq^* & z(q^* - 2)\mathbf{g} & z^2\mathbf{g} \\
 & & & & 1 & & & q^* & & & -2z\mathbf{g} & \\
 & & & & & & 1 & -q^* & & & 2z\mathbf{g} & \\
 & & & & & & & & 1 & -q^* & (q^* - 2)\mathbf{g} & 2z\mathbf{g}
\end{bmatrix}.
$$

The appearance of **f** and **g** demonstrates the analytic dependence on the parameters and explains the validity of analytic continuations.

Now we may solve B inductively on $w = (\bar{\beta}, d_2)$. The formulas are complicate and the details are thus omitted.

Theorem (LLQW-III 2013, CJM 2016)

The split assumption can be removed. The Quantum Leray–Hirsch Theorem for projective bundles and the analytic continuations of QH for ordinary flops hold without the splitting assumption on bundles.

END